Entanglement of formation for an arbitrary two-mode Gaussian state

Paulina Marian and Tudor A. Marian
Centre for Advanced Quantum Physics, University of Bucharest,
P.O.Box MG-11, R-077125 Bucharest-Măgurele, Romania
(Dated: December 16, 2008)

We write the optimal pure-state decomposition of any two-mode Gaussian state and show that its entanglement of formation coincides with the Gaussian one. This enables us to develop an insightful approach of evaluating the exact entanglement of formation. Its additivity is finally proven.

PACS numbers: 03.65.Ud, 03.67.Mn, 42.50.Dv

In recent years impressive efforts have been made to quantify the entanglement of two-party states of quantum systems. This trend was highly stimulated by the interest in exploiting entanglement as an efficient resource in quantum information processing. For any pure bipartite state a convenient measure of entanglement is now unanimously admitted, namely, the von Neumann entropy of its reduced states [1, 2]. Unlike the pure-state case, several measures of entanglement have been considered for mixed bipartite states on both finite- and infinite-dimensional Hilbert spaces [3]. Because of its operational meaning, the *entanglement of formation* (EF) of a mixed bipartite state, introduced by Bennett *et al.* [4], plays a significant role: it is the minimal amount of entanglement of any ensemble of pure bipartite states realizing the given state. To be explicit, the EF of a mixed bipartite state ρ is defined as an infimum taken over all its pure-state convex decompositions [4]:

$$E_F(\rho) := \inf\{\sum_k p_k E(|\Psi_k\rangle\langle\Psi_k|) \mid \rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|\}. \quad (1)$$

Here $E(|\Psi_k\rangle\langle\Psi_k|)$ is the amount of entanglement of the pure bipartite state $|\Psi_k\rangle$. According to definition (1), evaluating the EF is a hard task, even for special quantum states. However, analytic evaluations of the EF have been carried out in a few finite-dimensional cases: general two-qubit states [5], isotropic states [6], and Werner states [7].

In quantum information with continuous variables, two-mode Gaussian states (TMGSs) of the quantum radiation field are especially accessible from both theoretical and experimental standpoints. Their usefulness was recently reviewed in Refs. [8, 9]. So far, the only evaluation of the exact EF in an infinite-dimensional Hilbert space has been performed for symmetric TMGSs [10]. Moreover, the additivity of the EF has been proven in this case [11]. The Peres-Simon separability theorem [12] made it possible to use Gaussian measures of entanglement. Within a Gaussian approach, the reference set of states involved in the definition of any accepted entanglement measure is restricted to the subset of the Gaussian ones. Thus, following an earlier distance-type proposal for quantifying entanglement due to Vedral and co-workers [2], several Gaussian evaluations employing the relative entropy [13] or the Bures metric [14, 15, 16] have been performed. In Ref.[11], a *Gaussian entanglement of formation* (GEF) has been introduced for any inseparable TMGS by analyzing its optimal decomposition into pure TMGSs.

The aim of the present work is threefold. First, we build the appropriate decomposition, Eq. (1), of a TMGS ρ_G that allows us to show that its EF and GEF coincide. We thus answer an open problem in continuous-variable quantum information [10, 11]. Second, we give a more comprehensible approach to the problem of evaluating the GEF by use of covariance matrices (CMs). This enables us to write equations that yield, via the resulting optimal decomposition, an analytic solution for the EF in the general case. We also get explicit results in the most interesting special cases. Third, based on this approach, we prove the additivity of the EF for two-mode Gaussian states.

Before proceeding we recall several useful properties of TMGSs. For later convenience, we choose to describe any TMGS ρ_G by its characteristic function (CF),

$$\chi_G(\lambda_1, \lambda_2) := \mathrm{Tr}[\rho_G D_1(\lambda_1) D_2(\lambda_2)], \quad (2)$$

where $D(\alpha) := \exp(\alpha a^\dagger - \alpha^* a)$ is a Weyl displacement operator. The CF of an undisplaced TMGS is $\chi_G(x) = \exp\left(-\frac{1}{2}x^T \mathcal{V} x\right)$. Here $x \in \mathbb{R}^4$ and $\mathcal{V}$ is the real, symmetric, and positive 4×4 CM that completely describes the state. Its entries are the second-order moments of the canonical operators $q_j = (a_j + a_j^\dagger)/\sqrt{2}$, $p_j = (a_j - a_j^\dagger)/(\sqrt{2}i)$, where a_j and $a_j^\dagger$, $(j = 1, 2)$, are the amplitude operators of the modes. Note that $\mathcal{V} \in M_4(\mathbb{R})$ is the CM of a TMGS if and only if the Robertson-Schrödinger matrix inequality holds: $\mathcal{V} + \frac{i}{2}\Omega \geq 0$, $\Omega := i(\sigma_2 \oplus \sigma_2)$, with σ_2 a Pauli matrix. In particular, $\mathcal{D} := \det\left(\mathcal{V} + \frac{i}{2}\Omega\right) \geq 0$. Gaussian states whose CMs are connected by local symplectic transformations have the same *amount of entanglement* and belong to an equivalence class: their CMs are locally congruent to CMs having a *scaled standard form*,

$$\mathcal{V}(u_1, u_2) = \begin{pmatrix} b_1 u_1 & 0 & c\sqrt{u_1 u_2} & 0 \\ 0 & b_1/u_1 & 0 & d/\sqrt{u_1 u_2} \\ c\sqrt{u_1 u_2} & 0 & b_2 u_2 & 0 \\ 0 & d/\sqrt{u_1 u_2} & 0 & b_2/u_2 \end{pmatrix}. \quad (3)$$

In Eq. (3), $u_1 \geq 1, u_2 \geq 1$ are one-mode squeezing factors. The unscaled standard form $\mathcal{V}(1,1)$ of the CM, introduced in Ref.[17], is expressed in terms of four parameters $b_1,\ b_2,\ c,\ d$. They are local invariants and determine the entanglement properties of the whole equivalence class. Recall the locally invariant Peres-Simon separability condition for a TMGS [12], $\tilde{\mathcal{V}} + \frac{i}{2}\Omega \geq 0$, with $\tilde{\mathcal{V}}$ denoting the CM of the partially transposed density operator. This matrix inequality reduces to the Simon separability test [12]:

$$\tilde{\mathcal{D}} := \det\left(\tilde{\mathcal{V}} + \frac{i}{2}\Omega\right) = \det\mathcal{V} - \frac{1}{4}(b_1^2 + b_2^2 + 2c|d|) + \frac{1}{16} \geq 0. \tag{4}$$

The concept of classicality (existence of the Glauber-Sudarshan P representation of the density operator) is central in our present treatment of the EF. A TMGS with a CM (3) is classical if and only if the matrix $\mathcal{V}(u_1,u_2) - \frac{1}{2}I_4$ is non-negative, with I_4 the 4×4 identity matrix. This requirement is equivalent to the non-negativity of all its principal minors. Remark that the classicality conditions are not locally invariant, depending on the factors u_1, u_2.

We start on the programme of Eq. (1) for an inseparable mixed TMGS ρ_G, whose CM has a scaled standard form (3). Its four standard-form parameters $b_1, b_2, c \geq |d| = -d > 0$ are given, while the scaling factors u_1, u_2 are unknown. Continuous pure-state decompositions of such a state are convex combinations of the type

$$\rho_G = \int \mathrm{d}^2\beta_1 \mathrm{d}^2\beta_2 P(\beta_1,\beta_2)|\Psi(\beta_1,\beta_2)\rangle\langle\Psi(\beta_1,\beta_2)|. \tag{5}$$

$P(\beta_1,\beta_2)$ is a non-negative normalized distribution function and $|\Psi(\beta_1,\beta_2)\rangle$ is a state vector depending on the complex variables β_1, β_2. In accordance with the EF definition, Eq. (1), the pure states in the above continuous combination should achieve an optimal decomposition of the given state ρ_G. To this end, we make use of an important theorem regarding the ranking of entanglement among pure two-mode states proven in a recent paper of Giedke *et al.* [10]: For a given EPR uncertainty, the minimal entanglement over the whole class of pure states is reached by a Gaussian one, the two-mode squeezed vacuum state (TMSVS). This important result leads to the *key idea* of our treatment: Owing to the Gaussian nature of the two-mode state ρ_G, as well as to the scaled standard form (3) of its CM, we are allowed from the very beginning to restrict ourselves in Eq. (5) to equally entangled pure states obtained by displacing a unique TMSVS. Among all ensembles of such pure two-mode states that realize the given mixed state ρ_G we have to find the one possessing the minimal entanglement. Let us denote by $\rho_0 = |\Psi_0\rangle\langle\Psi_0|$ the TMSVS entering this optimal convex expansion:

$$\rho_G = \int \mathrm{d}^2\beta_1 \mathrm{d}^2\beta_2 P(\beta_1,\beta_2) D_1(\beta_1) D_2(\beta_2) \rho_0 D_2^\dagger(\beta_2) D_1^\dagger(\beta_1). \tag{6}$$

According to Eq. (6), the exact EF of the given mixed two-mode state ρ_G reduces to the amount of entanglement of the TMSVS ρ_0:

$$E_F(\rho_G) = E(\rho_0). \tag{7}$$

Recall now that a TMSVS is a Gaussian state whose CM has precisely the unscaled standard form $\mathcal{V}(1,1)$, Eq. (3): its parameters $b_1 = b_2 =: x > 1/2, c = -d =: y > 0$ are subjected to the purity condition

$$x^2 - y^2 = \frac{1}{4}. \tag{8}$$

The entanglement of a TMSVS is the von Neumann entropy of its one-mode reductions,

$$E(\rho_0) = (x + \frac{1}{2})\ln(x + \frac{1}{2}) - (x - \frac{1}{2})\ln(x - \frac{1}{2}), \tag{9}$$

which is an increasing and concave function of the variable $x > \frac{1}{2}$. We notice that the optimal convex decomposition (6) stands the pertinent test of the pure-state limit case: $\rho_G = \rho_0$, with $u_1 = u_2 = 1$ and $P(\beta_1,\beta_2) = \delta_2(\beta_1)\delta_2(\beta_2)$. To evaluate the EF (7), one has to be able to determine the optimal decomposition (6), *i.e.*, both the distribution function $P(\beta_1,\beta_2)$ and the TMSVS ρ_0. Provided that this can be effectively done for any TMGS, Eq. (6) displays the first result of our work: *the EF for a two-mode Gaussian state coincides with its GEF.*

We now take advantage of a fact well known in quantum optics: decompositions of the type (6) do have a clear meaning starting with Glauber's seminal work on the coherent states of the electromagnetic field [18]. Accordingly, Eq. (6) gives the density operator ρ_G of a superposition of two fields: one is in a classical state ρ_C having the regular Glauber-Sudarshan P representation $P(\beta_1,\beta_2)$, and the other is in the pure state ρ_0. By employing the corresponding CFs and writing the P representation as the Fourier transform of the normally-ordered CF, $\chi^{(N)}(\lambda_1,\lambda_2) := \exp\left(\frac{1}{2}(|\lambda_1|^2 + |\lambda_2|^2)\right)\chi(\lambda_1,\lambda_2)$, Eq. (6) leads to the multiplication law

$$\chi_G^{(N)}(\lambda_1,\lambda_2) = \chi_0^{(N)}(\lambda_1,\lambda_2)\chi_C^{(N)}(\lambda_1,\lambda_2), \tag{10}$$

with χ_C denoting the CF of the classical state ρ_C of the superposed field [19, 20]. It follows that the CF $\chi_C(\lambda_1,\lambda_2)$ is also Gaussian. Equation (10) results in an addition rule for the CMs of the Gaussian states involved:

$$\mathcal{V}(u_1,u_2) = \mathcal{V}_0 + \mathcal{V}_C - \frac{1}{2}I_4. \tag{11}$$

Our method towards finding the optimal pure-state decomposition concentrates on the properties of the classical state ρ_C. We first show that ρ_C belongs to the boundary $\partial\mathcal{P}$ of the set $\mathcal{P}$ of all classical TMGSs (*Property 1*). Then we prove that ρ_C is also on the boundary $\partial\mathcal{S}$ of the larger set $\mathcal{S}$ of all separable TMGSs: $\mathcal{S} \supset \mathcal{P}$ (*Property 2*), Refs. [21, 22].

Property 1: The superposed classical state ρ_C is at the classicality threshold. Indeed, for the optimal superposition, the CM $\mathcal{V}(u_1, u_2)$ should be as close as possible to $\mathcal{V}_0$. This happens when the principal minors of rank 3 and 4 of the non-negative matrix $\mathcal{V}(u_1, u_2) - \mathcal{V}_0 = \mathcal{V}_C - \frac{1}{2}I_4$ are zero. By the same token, the Gaussian state ρ_C is at the border of classicality $\partial\mathcal{P}$. Explicitly, the condition $\det(\mathcal{V}_C - \frac{1}{2}I_4) = 0$ holds with the left-hand side expressed as a product of two vanishing factors:

$$(b_1u_1 - x)(b_2u_2 - x) - (c\sqrt{u_1u_2} - y)^2 = 0, \qquad (12)$$

$$(b_1/u_1 - x)(b_2/u_2 - x) - (|d|/\sqrt{u_1u_2} - y)^2 = 0. \quad (13)$$

Equations (12) and (13) are in agreement with the Gaussian optimality conditions written in the pioneering work Ref.[11] on different grounds. Making use of Eqs. (8), (12), and (13), we can impose to the one-variable function $x = x(u_1, u_2(u_1))$ the minimization condition $\frac{\mathrm{d}x}{\mathrm{d}u_1} = 0$. We get therefore a fourth independent algebraic equation,

$$\frac{b_1u_1 - x}{b_1/u_1 - x} = \frac{b_2u_2 - x}{b_2/u_2 - x}, \qquad (14)$$

which implies an additional property of the state ρ_C.

Property 2: The superposed classical state ρ_C is at the separability limit as well. To prove this statement, we use Eqs. (8), (12), and (13) to evaluate the Simon invariant $\tilde{\mathcal{D}}$, Eq. (4), of the Gaussian state ρ_C. Taking into account Eq. (14), we get $\tilde{\mathcal{D}} = 0$, *i. e.*, $\rho_C \in \partial\mathcal{S}$.

The evaluation of the required EF reduces to solving a system of four non-linear algebraic equations, namely, Eqs. (8), and (12)– (14), with four unknowns: u_1, u_2, x, y. Let us denote its solution by w_1, w_2, x_m, y_m. The above algebraic system yields a quartic equation, $\sum_{n=0}^{4} \mathcal{A}_n p^n = 0$, for the product $p := u_1u_2$. The coefficients $\mathcal{A}_n$ are quite simple polynomials in the four standard-form parameters of the given inseparable TMGS:

$$\mathcal{A}_0 = (b_1b_2 - d^2)\left[b_1(b_1b_2 - d^2) - \frac{b_2}{4}\right] \times \left[b_2(b_1b_2 - d^2) - \frac{b_1}{4}\right] > 0,$$

$$\mathcal{A}_1 = -[c(b_1b_2 - d^2) + \frac{|d|}{4}]\left\{(b_1 - b_2)^2[c(b_1b_2 - d^2) + \frac{|d|}{4}] + 2b_1b_2(c - |d|)\left(b_1b_2 - d^2 - \frac{1}{4}\right)\right\} \leq 0,$$

$$\mathcal{A}_2 = [(b_1c - b_2|d|)(b_1|d| - b_2c) + c|d|\mathcal{Z}](\det\mathcal{V} + 1/16) - 2(b_1^2b_2^2 - c^2d^2)\mathcal{D} - c|d|\det\mathcal{V},$$

$$\mathcal{A}_3 = \mathcal{A}_1(c \leftrightarrow |d|), \ \mathcal{A}_4 = \mathcal{A}_0(c \leftrightarrow |d|) \geq 0. \qquad (15)$$

We have introduced the symplectic invariant $\mathcal{Z} := b_1^2 + b_2^2 + 2cd \geq 1/2$. At $p = 1$, the above quartic polynomial has a negative value, except for $c = |d|$, when it vanishes. This implies the existence of a convenient root $p_m = w_1w_2 \geq 1$ for any inseparable mixed TMGS. Had we got p_m, it could be used to obtain the optimal y_m as the smallest root of a quadratic trinomial $\mathcal{B}_2(p)y^2 + \mathcal{B}_1(p)y + \mathcal{B}_0(p)$ whose coefficients,

$$\mathcal{B}_0(p) = -\tilde{\mathcal{D}}p \geq 0,$$

$$\mathcal{B}_1(p) = -2\sqrt{p}\left(\left[|d|(b_1b_2 - c^2) + c/4\right]p + \left[c(b_1b_2 - d^2) + |d|/4\right]\right) < 0,$$

$$\mathcal{B}_2(p) = (b_1b_2 - c^2)p^2 + \mathcal{Z}p + (b_1b_2 - d^2) > 0, \quad (16)$$

are evaluated at $p = p_m$. We mention that in four significant particular cases (defined by special relations between standard-form parameters) we have found simple solutions by direct use of Eqs. (12)– (14). We have then recovered them by exploiting Eqs. (15) and (16).

As a first salient example, we consider an entangled symmetric TMGS, whose standard-form parameters are $b_1 = b_2 =: b, \ \ c \geq |d| = -d > 0$. The smallest symplectic eigenvalue $\tilde{\kappa}_-$ of the CM for the partially transposed density operator is in this case $\tilde{\kappa}_- = \sqrt{(b-c)(b-|d|)}$. In agreement with the results of the remarkable work Ref.[10], Eqs. (12)– (14) and (8) give:

$$w_1 = w_2 = \sqrt{\frac{b - |d|}{b - c}}, \quad x_m = \frac{\tilde{\kappa}_-^2 + 1/4}{2\tilde{\kappa}_-}.$$

A second class of notable bipartite states is that of two-mode squeezed thermal states. The standard-form parameters of such a state are $b_1 \geq b_2, \ \ c = |d| = -d > 0$. This case was considered previously in Refs. [23, 24], where the prescription of Ref.[11] to evaluate the GEF was followed. From our results,

$$w_1 = w_2 = 1, \ \ x_m = \frac{(b_1 + b_2)(b_1b_2 - c^2 + 1/4) - 2c\sqrt{\mathcal{D}}}{(b_1 + b_2)^2 - 4c^2},$$

one can see that x_m is not determined only by the eigenvalue $\tilde{\kappa}_- = \frac{1}{2}[b_1 + b_2 - \sqrt{(b_1 - b_2)^2 + 4c^2}]$.

A third example is that of a TMGS at the separability boundary: $\tilde{\mathcal{D}} = 0 \Longleftrightarrow \tilde{\kappa}_- = \frac{1}{2}$. We get $x_m = \frac{1}{2}, \quad y_m = 0$ and the optimal squeeze factors

$$w_1 = \sqrt{\frac{b_2(b_1b_2 - d^2) - \frac{1}{4}b_1}{b_2(b_1b_2 - c^2) - \frac{1}{4}b_1}}, \ \ w_2 = w_1(b_1 \leftrightarrow b_2).$$

Equation (6) becomes now the P representation of a state ρ_G at the border of classicality $\partial\mathcal{P}$ and that of separability $\partial\mathcal{S}$ as well.

A fourth class of entangled states consists of those TMGSs whose CMs have the smallest symplectic eigenvalue κ_-: $\mathcal{D}=0 \Longleftrightarrow \kappa_-=\frac{1}{2}$. These states were studied as having minimal negativity at fixed local and global purities [25]. Assuming that $b_1 \geq b_2, \;\; c \geq |d| = -d > 0$, we found two distinct solutions required by the sign of the difference $b_2c - b_1|d|$:

$$b_2c-b_1|d|<0: \quad x_m=\frac{b_1^2-b_2^2}{8(\det\mathcal{V}-\frac{1}{16})},$$

$$w_1=\sqrt{\frac{b_2(b_1b_2-d^2)-\frac{1}{4}b_1}{b_2(b_1b_2-c^2)-\frac{1}{4}b_1}}, \;\; w_2=w_1(b_1\leftrightarrow b_2).$$

$$b_2c-b_1|d|\geq 0: \;\; x_m=\frac{1}{2}\sqrt{\frac{b_1b_2}{b_1b_2-d^2}},$$

$$w_1=2\sqrt{\frac{b_1}{b_2}(b_1b_2-d^2)}, \;\; w_2=w_1(b_1\leftrightarrow b_2).$$

The above formulae for x_m are in agreement with those derived in other parametrization in Ref.[24], which follows the methods of Ref.[11].

The last issue we are here interested in is the additivity of the EF for TMGSs. Our present approach gives a straightforward answer to this open question [3]. We consider a four-mode product state $\rho_G\otimes\sigma_G$, where ρ_G and σ_G are entangled TMGSs. We denote the minimally entangled TMSVSs entering the optimal decompositions of the type (6) for both factors by ρ_0 and σ_0, respectively. Therefore, their tensor product $\rho_0\otimes\sigma_0$ enters the optimal convex decomposition of the four-mode state $\rho_G\otimes\sigma_G$. It follows the identity $E_F(\rho_G\otimes\sigma_G)=E(\rho_0\otimes\sigma_0)$. The well-known additivity property of the von Neumann entropy, $E(\rho_0\otimes\sigma_0)=E(\rho_0)+E(\sigma_0)$, yields the additivity of the EF for TMGSs:

$$E_F(\rho_G\otimes\sigma_G)=E_F(\rho_G)+E_F(\sigma_G).$$

Consequences of this property on evaluating other measures of entanglement are largely discussed in Ref.[3].

To sum up, we have reformulated the problem of evaluating the EF for TMGSs in terms of CFs and CMs. We have shown that the exact EF of such a state coincides with its Gaussian one. Although an analytic solution in the general case seems to be complicated, it can be found, nevertheless, by solving a quartic equation. Our general treatment allowed us to retrieve readily previous explicit results in some relevant particular cases. Based on our approach, we have finally proven the additivity of the EF for two-mode Gaussian states.

This work was supported by the Romanian Ministry of Education and Research through Grant No. IDEI-995/2007 for the University of Bucharest.

Note added.-During the completion of this Letter, an interesting treatment of the EF for a TMGS was given in Ref.[26]. Its relation to our present work will be discussed elsewhere.

[1] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K.Wootters, Phys. Rev. Lett. **76**, 722 (1996); C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).

[2] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. **78**, 2275 (1997); V. Vedral, M. B. Plenio, Phys. Rev. A **57**, 1619 (1998).

[3] M. B. Plenio and S. Virmani, Quantum Information and Computation **7**, 001 (2007).

[4] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K.Wootters, Phys. Rev. A **54**, 3824 (1996).

[5] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

[6] Barbara M. Terhal and K. G. H. Vollbrecht, Phys. Rev. Lett. **85**, 2625 (2000).

[7] K. G. H. Vollbrecht and R. F. Werner, Phys. Rev. A **64**, 062307 (2001).

[8] J. Eisert and M. B. Plenio, Int. J. Quantum Inf. **1**, 479 (2003); S. L. Braunstein and P. Van Loock, Rev. Mod. Phys. **77**, 513 (2005).

[9] A. Ferraro, S. Olivares, and M. G. A. Paris, *Gaussian States in Quantum Information* (Bibliopolis, Napoli, 2005); F. Dell'Anno, S. De Siena, and F. Illuminati, Phys. Rep. **428**, 53 (2006); G. Adesso and F. Illuminati, J. Phys. A: Math. Theor. **40**, 7821 (2007).

[10] G. Giedke, M. M. Wolf, O. Krüger, R. F. Werner, and J. I. Cirac, Phys. Rev. Lett. **91**, 107901 (2003).

[11] M. M. Wolf, G. Giedke, O. Krüger, R. F. Werner, and J. I. Cirac, Phys. Rev. A **69**, 052320 (2004).

[12] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000).

[13] S. Scheel and D.-G. Welsch, Phys. Rev. A **64**, 063811 (2001).

[14] Paulina Marian, T. A. Marian, and H. Scutaru, Phys. Rev. A **68**, 062309 (2003).

[15] Paulina Marian and T. A. Marian, Phys. Rev. A **77**, 062319 (2008).

[16] Paulina Marian and T. A. Marian, Eur. Phys. J. Special Topics **160**, 281 (2008).

[17] L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000).

[18] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).

[19] Paulina Marian and T. A. Marian, J. Phys. A: Math. Gen. **29**, 6233 (1996).

[20] Equation (10) can be obviously generalized to arbitrary multimode states. The only validity condition is the classicality of the state ρ_C. In Ref.[18], only superpositions of classical fields were considered and the convolution law of the corresponding well-behaved P representations was derived. In our paper [19], we pointed out the multiplication law of the CFs and employed it to analyze the influence of thermal noise on nonclassical properties of one-mode states.

[21] B. G. Englert and K. Wódkiewicz, Phys. Rev. A **65**, 054303 (2002).
[22] M. C. de Oliveira, Phys. Rev. A **70**, 034303 (2004).
[23] Li-Zhen Jiang, Int. J. Quantum Inf. **2**, 273 (2004).
[24] G. Adesso and F. Illuminati, Phys. Rev. A **72**, 032334 (2005).
[25] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. Lett.**92**, 087901 (2004); G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. A **70**, 022318 (2004).
[26] J. Solomon Ivan and R. Simon, arXiv:0808.1658.